Journal of Applied Mathematics and Mechanics 72 (2008) 159–163

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

Stabilization of the equilibria of dynamical systems☆

A.P. Blinov

Moscow, Russia

ARTICLE INFO

Article history:
Received 25 January 2007

ABSTRACT

Based on Rumyantsev's method, a procedure is developed for stabilizing stable and unstable equilibria of dynamical systems by continuous and modulus-constrained control actions. It is shown that, when modulus constraints are imposed on the controls, and when the quadratic-form coefficients are reduced in modulus, the optimal stabilization, in terms of this method, approximates to time-optimal stabilization. A solution is obtained of the problem of stabilizing unstable equilibrium positions at which the potential energy of the system has neither a maximum nor a minimum (or, in particular, at which the potential energy is identically equal to zero).

Rumyantsev's method[1] enables us to achieve constructive stabilization of the stable steady motion of a system by additional forces, when a certain functional characterizing the quality of the control is minimized. It has been shown[2,3] that the stabilizing properties of controls constructed by this method, with modulus constraints imposed on them, are retained for a certain class of controlled systems; the area of controllability was estimated.

New methods for controlling non-linear mechanical systems[4] guarantee (even under conditions of uncertainty) that the system is brought to the origin of coordinates in a finite time with any form of perturbations, provided the modulus of the latter does not exceed the admissible controls. However, under conditions of complete determinacy, to reduce the level of control actions, it is useful to take into account not only the modulus of perturbations but also their signs.

When piecewise linear control algorithms with a step change in feedback factors are implemented,[4] there are shock effects on the system, and the phase trajectory during stabilization does not necessarily intersect the levels of the Lyapunov function, i.e., it cannot be guaranteed that there will be no "overshoots" of the phase point from the initial region.

1. The stabilization of stable equilibria

Suppose that, for a dynamical system

$$\dot{x} = X(x), \quad X(0) = 0, \quad x \in R^n \tag{1.1}$$

in the region $G \in R^n$, the conditions of existence and uniqueness of the solutions are satisfied, and a positive-definite Lyapunov function V is known such that its complete time derivative W, by virtue of system (1.1), is sign- negative or identically equal to zero.

Suppose that, to stabilize the steady motion, a control $u \in R^r$ is introduced into system (1.1) such that it takes the form[1]

$$\dot{x}_s = X_s(x) + \sum_{j=1}^{r} m_{sj}(x)u_j \tag{1.2}$$

and optimal controls are found

$$u_j^0 = -\frac{1}{2}\sum_{k=1}^{r}\frac{\Delta_{kj}}{\Delta}\sum_{i=1}^{n}\frac{\partial V}{\partial x_i}m_{ik}; \quad \Delta = \det(\beta_{ij}) \tag{1.3}$$

☆ *Prikl. Mat. Mekh.* Vol. 72, No. 2, pp. 260–265, 2008.

0021-8928/$ – see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jappmathmech.2008.04.009

(Δ_{kj} is the cofactor of the element β_{ij}) with the quality criterion

$$I = \int_{t_0}^{\infty}[F(x)+K]dt; \quad K = \sum_{i,j=1}^{r}\beta_{ij}u_iu_j$$

where K is a specified positive-definite quadratic form with symmetric coefficients, and $F(x)$ is a non-negative function that is chosen depending on the coefficients β_{ij}.[1]

It may turn out that, in region G, the controls found, u_j^0, cannot be realized on account of modulus constraints $|u_j| \le a$. Then, in system (1.2), we will change to new controls υ_j and, using a certain arbitrariness of choice of the coefficients β_{ij}, we will replace them with

$$\beta'_{ij} = \frac{b}{a}\beta_{ij}; \quad b = \sup_{G,j}|u_j^0|, \quad a < b$$

The new optimal controls υ_j^0 will differ from the former controls u_j^0 in that, infront of the ratio Δ_{kj}/Δ there will be a factor a/b, i.e., $\upsilon_j^0 = (a/b)u_j^0$ are the new optimal controls.

Thus, the optimal control extends over the entire region G (by the double-sweep method). Here, by virtue of system (1.2), the total time derivative of the Lyapunov function

$$\dot{V} = W - 2K^0, \quad K^0 = \sum_{i,j=1}^{r}\beta_{ij}u_i^0u_j^0$$

on changing to new controls υ_j^0 (with $W \equiv 0$) decreases by a factor of a/b, i.e. stabilization is more sluggish. (Here and below it is assumed that the manifold $\dot{V} = 0$ does not contain entire trajectories (1.2).)

The following question arises: is it impossible for modulus-constrained controls (perhaps, non-optimal controls in the entire region G) to accelerate the motion of the phase point towards the equilibrium position?

We will return to the former controls u_j with modulus constraints

$$|u_j| \le a_j, \quad j = 1, 2, \dots, r$$

In phase space we will fix the regions

$$G_j = \{x: |u_j^0| \le a_j\}, \quad j = 1, 2, \dots, r$$

and define the new controls

$$u_j = \begin{cases} u_j^0, & \text{if } x \in G_j \\ a_j \operatorname{sign} u_j^0, & \text{if } x \in G_j \end{cases} \tag{1.4}$$

With control actions of this type, the right-hand sides of Eq. (1.2) are continuous in region G, and therefore, by virtue of Eq. (1.2), the time derivative of the Lyapunov function is defined and continuous in this region G and has the form

$$\dot{V} = W + \sum_{s=1}^{n}\frac{\partial V}{\partial x_s}\sum_{j=1}^{r}m_{sj}u_j \tag{1.5}$$

As before, this derivative will be sign negative (even when $W \equiv 0$), and the manifold on which it may become zero remains the same as in control without constraints. In fact, in the worst case (when $W \equiv 0$), the derivative $\dot{V}$ vanishes only when all controls simultaneously become zero.

In fact, if it is assumed that $\dot{V}$ can become zero for certain $u_k \ne 0$ and the remainder $u_j = 0, j \ne k$, then the corresponding phase point $(x_{1k}, \dots, x_{nk})$ lies either within the region G_k or outside G_k. In the latter case, $\dot{V}$ has the same sign as $\dot{V}$ by virtue of a system with optimal control, i.e., the sign will be negative. In the former case, however, when the phase point lies within the region G_k, it will simultaneously lie within the region $G_0 \bigcap_j G_j$ and obey the optimal control. However, for the optimal control we have the expression

$$\dot{V} = -K^0$$

which vanishes only when all the controls simultaneously become zero.

It follows that, the lower the modulus of the coefficients β_{ij} chosen, the smaller will be the region G_0, i.e., the control process approximates to a discontinuous process reminiscent of time-optimal control.

To illustrate this, consider the problem[5] of the fastest motion of the system to the origin of coordinates.

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u; \quad |u| \le 1$$

Such motion is ensured by the control

$$u = \operatorname{sign}(\sin(t - \alpha_0)), \quad \alpha_0 = \text{const}$$

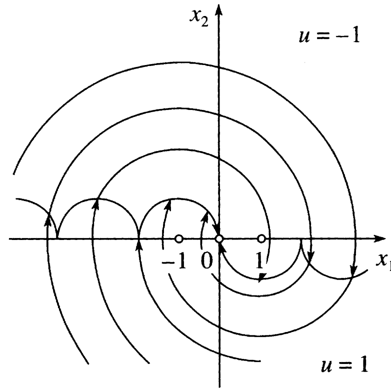

Fig. 1.

for which the phase trajectories consist of arcs of circles with centres at the points (−1, 0), (1, 0), while the lines of the switching of controls consist of arcs of circles of unit radius with centres on the axis $x_1 : x_1 = 1 \pm 2k$ ($k = 0, 1, 2, \ldots$) (Fig. 1).

Applying the method of stabilization[1] to this system with the quality functional

$$I = \int_0^\infty [F(x) + \beta u^2] dt$$

and with the constraint $|u| \le 1$, outside the strip $|x_2| \le 2\beta$ we will obtain the same phase trajectories (arcs of circles), while within the strip we will obtain pieces of Rumyantsev- optimal smooth phase trajectories (Fig. 2).

Such a combined control, inferior to time-optimal control, has an advantage in implementation inasmuch as, because of the drive time lag, discontinuous controls cannot be realized accurately, and shock effects create large overloads in the controlled system.

2. Stabilization of unstable equilibria

Earlier work[3] showed the possibility of stabilizing unstable equilibrium positions for a certain class of dynamical systems by controls consisting of two components, one of which ensures the stability of these equilibria, and the other ensures optimal stabilization by Rumyantsev's method.[1] It is obvious that, in this case also, the second control component can be assumed to be constrained in modulus.

The above-mentioned class of systems includes conservative systems with equilibrium positions at which the potential energy has a maximum. Since the stabilization of unstable equilibrium positions at which the potential energy has neither a maximum nor a minimum is of interest, we will consider this case.

Suppose the dynamical controlled system, written in the form of Lagrange's equations (and, if there are cyclic coordinates, in the form of Routh's equations)

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = -\frac{\partial \Pi}{\partial q_i} + \sum_{j=1}^{n} F_{ij} u_j, \quad i = 1, 2, \ldots, n \tag{2.1}$$

has the unstable equilibrium position

$$q = 0, \quad \dot{q} = 0, \quad q = (q_1, q_2, \ldots, q_n), \quad u \equiv 0, \quad u = (u_1, u_2, \ldots, u_n)$$

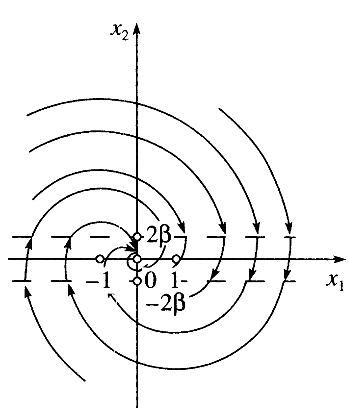

Fig. 2.

and the quadratic part of the expansion of the potential energy Π is not sign-definite, but its quadratic part

$$\Pi_2 = \sum_{i,j=1}^{n} a_{ij}q_iq_j, \quad a_{ij} = \left.\frac{\partial^2\Pi}{2\partial q_i\partial q_j}\right|_{q=0} \tag{2.2}$$

is positive-definite only for $q_1, q_2, \ldots, q_{k-1}$.

We will define the function

$$\begin{aligned} P(q) &= P_{1\,k}q_1q_k + \ldots + P_{k-1\,k}q_{k-1}q_k + \frac{1}{2}P_{k\,k}q_k^2 + P_{0\,k}(q_k) + \\ &+ P_{1\,k+1}q_1q_{k+1} + \ldots + P_{k\,k+1}q_kq_{k+1} + \frac{1}{2}P_{k+1\,k+1}q_{k+1}^2 + P_{0\,k+1}(q_{k+1}) + \ldots + \\ &+ P_{1n}q_1q_n + \ldots + P_{n-1\,n}q_{n-1}q_n + \frac{1}{2}P_{n\,n}q_n^2 + P_{0\,n}(q_n) \end{aligned} \tag{2.3}$$

in which the terms $P_{0l}(q_l)$ are even non-negative functions and

$$P_{0\,l}(0) = 0, \quad P_{l\,l} = \begin{cases} -a_{ll} + \varepsilon_l, & \varepsilon_l > 0, \ \text{if } a_{ll} \le 0 \\ 0, & \text{if } a_{ll} > 0 \end{cases}, \quad P_{lr} = -a_{lr} \ \text{When } l \ne r$$

Then the sum $\Pi + P$ will be a positive-definite function in the vicinity of $q = 0$. Through the choice of functions $P_{0l}(q_l)$, and, perhaps, through a different choice of coefficients P_{lr} to that made earlier, this vicinity D (a potential well) can be extended.

We will assume that

$$u_j = w_j + \upsilon_j, \quad \sum_{j=1}^{n} F_{i\,j}w_j = \frac{\partial P}{\partial q_i}, \quad i = 1, 2, \ldots, n$$

From the system of equations obtained, with $\det(F_{ij}) \ne 0$, the controls w_j, $w_j(0) = 0$ are defined uniquely, apart from the choice of the coefficients ε_l, which can be determined from the constraint $|w_j| \le b_j$ when $q \in D$.

The function $V = T + \Pi + P$ is a positive-definite Lyapunov function. Its total time derivative is zero, by virtue of the system controlled by w_j, and therefore, by controls υ_j, the optimal or combined stabilization of the equilibrium $q = 0$, $\dot{q} = 0$ can be realized.

Example. Consider a double pendulum. Suppose m_1 and m_2 are the masses of the pendulums, l_1 and l_2 are the lengths of the pendulums and f_1 and f_2 are the angles of deflection of the pendulums from the vertical. The potential energy of the double pendulum has the form

$$\Pi = g_1(1 - \cos\varphi_1) + g_2(1 - \cos\varphi_2); \quad g_1 = (m_1 + m_2)gl_1, \quad g_2 = m_2gl_2.$$

The equilibrium position

$$\varphi_1 = 0, \quad \varphi_2 = \pi, \quad \dot{\varphi}_1 = \dot{\varphi}_2 = 0$$

is unstable. Putting $\varphi_2 = \psi + \pi$, we expand the function Π in the neighbourhood of zero

$$\Pi = g_1\left(\frac{1}{2}\varphi_1^2 - \frac{1}{24}\varphi_1^4 + \ldots\right) + g_2\left(2 - \frac{1}{2}\psi^2 + \frac{1}{24}\psi^4 - \ldots\right)$$

Chossing

$$P_0 = 0, \quad P_1 = 0, \quad P_2 = g_2(1 + \varepsilon)$$

we obtain a matrix of the coefficients of the quadratic form U of the form $\mathrm{diag}(g_1, \varepsilon g_2)$. Suppose w_2 is the control moment applied to the lower pendulum in the moving joint; then $w_1 - w_2$ is the control moment applied to the upper pendulum in the stationary joint. Now, from system (2.2) we find

$$w_2 = w_1 = -g_2(1 + \varepsilon)\psi$$

The same type of instability occurs for a point mass positioned at the collinear libration point L_2. The point mass, being slightly displaced from the point L_2 perpendicular to the segment connecting the gravitating bodies, is attracted to the point L_2, but, being displaced from L_2 along the segment, it moves away from point L_2 as time passes. Thus, the conditions of stabilization considered above are satisfied. This enables us, for example, to improve earlier results[6] by changing from discontinuous controls retaining the centre of mass of a spacecraft in the vicinity of the photogravitational libration point L_2^*, to continuous stabilization of the position of the centre of mass at this point.

3. The problem of minimizing energy consumption

We will consider in more detail the problem of minimizing the energy consumption when stabilizing the equilibrium of a system by choosing the coefficients P_{lk} occurring in the quadratic part of the function $P(q)$.

Suppose for the moment that $n=k$. In the notation of the coefficients P_{lk}, for convenience, the second subscript will be omitted (i.e. $P_{lk}=P_l$). We will require that the determinant

$$\Delta_k = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1k-1} & x_1 \\ a_{21} & a_{22} & \dots & a_{k-1\,1} & x_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{k-11} & a_{k-12} & \dots & a_{k-1\,k-1} & x_{k-1} \\ x_1 & x_2 & \dots & x_{k-1} & x_k \end{vmatrix}; \quad x_i = a_{ik}+P_i$$

vanishes.

According to the rule for the expansion of a bordered determinant, we can write

$$\Delta_k = \Delta_{k-1}x_k - \sum_{i,j=1}^{k-1} A_{ij}x_ix_j = \Delta_{k-1}(a_{kk}+P_k) - \sum_{i=l}^{k-1} A_{ii}(a_{ik}+P_i)^2 -$$

$$-2\sum_{1\le i\le j\le k-1} A_{ij}(a_{ik}+P_i)(a_{jk}+P_j); \quad \Delta_{k-1} = \det(a_{ij})>0,$$

where A_{ij} is the cofactor of the element a_{ij}.

Requiring that the sum

$$S = P_1^2+P_2^2+\ldots+P_k^2$$

be a minimum, and putting $\Phi=S-\lambda\Delta_k$, where λ is the Lagrange multiplier, we will write the necessary conditions for an extremum of the function Φ

$$P_1+\lambda A_{11}(a_{ik}+P_l)+\lambda\sum_{j=2}^{k-1} A_{1j}(a_{jk}+P_j) = 0, \ldots,$$

$$P_l+\lambda A_{ll}(a_{lk}+P_l)+\lambda\sum_{i=1,\,i\ne l}^{k-1} A_{il}(a_{ik}+P_i) = 0, \ldots$$

$$\ldots, P_{k-1}+\lambda A_{k-1\,k-1}(a_{k-1\,k}+P_{k-1}) + \lambda\sum_{i=1}^{k-2} A_{i\,k-1}(a_{ik}+P_i) = 0, \quad P_k-\frac{1}{2}\lambda\Delta_{k-1} = 0$$

This system of equations, at least at sufficiently small $|\lambda|$, has a unique solution, and the quantities P_i are expressed rationally in terms of λ.

After substituting P_i into the equation $\Delta_k=0$, an equation of degree $2k-1$ in λ is obtained, and therefore at least one real solution exists. The values of P_i which make the sum S a minimum are also determined from the solutions obtained.

In order for the determinant Δ_k to remain positive, it is sufficient to add to the value of P_k found as low a value $\varepsilon>0$ as desired.

If the dimension of the stabilized system $n>k$, then the procedure considered for reducing the quadratic form to a positive-definite form must be repeated with the determinant Δ_{k+1}, etc., up to Δ_n.

Thus, the procedure for the combined stabilization of the stable equilibria of dynamical systems, based on Rumyantsev's method, is extended to the stabilization of unstable equilibria.

References

1. Rumyantsev VV. The optimal stabilization of controlled systems. *Prikl Mat Mekh* 1970;**34**(3):440–56.
2. Blinov AP. Optimal stabilization of controlled systems. *Prikl Mat Mekh* 1982;**46**(3):366–73.
3. Blinov AP. Estimation of the region of controllability in non-linear systems. *Prikl Mat Mekh* 1984;**48**(4):593–600.
4. Chernous'ko FL, Anan'yevskii IM, Reshmin SA. *Methods for Controlling Non-linear Mechanical Systems*. Moscow: Fizmatlit; 2006.
5. Pontryagin LS, Boltyanskii VG, Gamkrelidze RV, Mishchenko EF. *The Mathematical Theory of Optimal Processes*. New York.
6. Blinov AP. The optimal stabilization of steady motion and the reorientation of a spacecraft by light-pressure forces. *Kosmich Issled* 1983;**21**(5):675–84.

Translated by P.S.C.